PG – 213

IV Semester M.Sc. Degree Examination, June 2017
(CBCS)
MATHEMATICS
M402 TC : Theory of Numbers

Time : 3 Hours Max. Marks : 70

Instructions : 1) *Answer **any five full** questions.*
2) ***All** questions carry **equal** marks.*

1. a) Define the Mobius function $\mu(n)$. Prove that $\sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1 \end{cases}$ 4

b) Define the Euler's totient function $\varphi(n)$. Prove that $\varphi(n) = \sum_{d|n}\mu(d)\frac{n}{d}$. 6

c) Find all integers n such that $\varphi(n) = \varphi(2n)$. 4

2. a) For $n \geq 1$, prove that $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$. 5

b) If $a \mid b$, prove that $\varphi(a) \mid \varphi(b)$. 4

c) For $n \geq 1$, prove that $\Lambda(n) = \sum_{d|n}\mu(d)\log\frac{n}{d} = -\sum_{d|n}\mu(d)\log d$. 5

3. a) When a number theoretic function is called multiplicative ? If f and g are multiplicative, prove that their Dirichlet product $f * g$ is also multiplicative. 4

b) If both g and $f * g$ are multiplicative, prove that f is also multiplicative. 6

c) Let f be multiplicative, g be any arithmetical function and that $f(p^{n+1}) = f(p)\, f(p^n) - g(p)\, f(p^{n-1})$, for all primes p and all $n \geq 1$. Then prove that for each prime p, the Bell series for f has the form

$$f_p(x) = \frac{1}{1 - f(p)x + g(p)x^2}$$ 4

P.T.O.

4. a) If $(a, m) = 1$, prove that $a^{\varphi(m)} \equiv 1 \pmod m$. 4

 b) Given a prime p, let $f(x) = c_0 + c_1x + \ldots + c_nx^n$ be a polynomial of degree n with integer coefficients such that $c_n \not\equiv 0 \pmod p$. Prove that the polynomial congruence $f(x) \equiv 0 \pmod p$ has at most n solutions. 6

 c) Prove that $5n^3 + 7n^5 \equiv 0 \pmod{12}$ for all integers n. 4

5. a) Let $n > 1$. Prove that $(n-1)! \equiv -1 \pmod n$ if and only if n is a prime. 5

 b) State and prove the Chinese remainder theorem. 6

 c) Find all x which simultaneously satisfy the system of congruences

 $x \equiv 1 \pmod 3$,

 $x \equiv 2 \pmod 4$,

 $x \equiv 3 \pmod 5$. 3

6. a) Define the Legendre's symbol $(n \mid p)$. Let p be an odd prime. Then for all n, prove that $(n \mid p) \equiv n^{\frac{(p-1)}{2}} \pmod p$. 4

 b) If p and q are distinct odd primes, prove that $(p \mid q)(q \mid p) = (-1)^{(p-1)(q-1)/4}$. 6

 c) Compute $(888 \mid 1999)$. 4

7. a) If $\alpha \geq 3$, prove that there are no primitive roots mod 2^{α}. 5

 b) Let g be a primitive root mod p such that $g^{p-1} \not\equiv 1 \pmod{p^2}$. Then prove that for every $\alpha \geq 2$, $g^{\varphi(p^{\alpha-1})} \not\equiv 1 \pmod{p^{\alpha}}$. 5

 c) Prove that 3 is a primitive root mod p if p is a prime of the form $2^n + 1$, $n > 1$. 4

8. a) Prove that

 i) $(1 + x)(1 + x^3)(1 + x^5)\ldots$

$$= 1 + \frac{x}{(1-x^2)} + \frac{x^4}{(1-x^2)(1-x^4)} + \frac{x^9}{(1-x^2)(1-x^4)(1-x^6)} + \ldots$$

 ii) $(1 + x^2)(1 + x^4)(1 + x^6)\ldots$

$$= 1 + \frac{x^2}{(1-x^2)} + \frac{x^6}{(1-x^2)(1-x^4)} + \frac{x^{12}}{(1-x^2)(1-x^4)(1-x^6)} + \ldots$$ 4

 b) State and prove the Jacobi's triple product identity. 10